A Model equilibrium: proofs

Proof of Proposition 1
The potential is a function Q from the space of actions to the real line such that $Q\left(g_{ij}, g_{-ij}, X\right) - Q\left(g'_{ij}, g_{-ij}, X\right) = U_i\left(g_{ij}, g_{-ij}, X\right) - U_i\left(g'_{ij}, g_{-ij}, X\right)$, for any ij.[45] A simple computation shows that, for any ij

$$\begin{aligned} Q\left(g_{ij}=1, g_{-ij}, X\right) - Q\left(g_{ij}=0, g_{-ij}, X\right) &= u_{ij} + g_{ji}m_{ij} + \sum_{\substack{k=1\\k\neq i,j}}^{n} g_{jk}v_{ik} + \sum_{\substack{k=1\\k\neq i,j}}^{n} g_{ki}v_{kj} \\ &= U_i\left(g_{ij}=1, g_{-ij}, X\right) - U_i\left(g_{ij}=0, g_{-ij}, X\right) \end{aligned}$$

therefore Q is the potential of the network formation game.

Proof of Corollary 1
The proof consists of showing that $Q\left(g, X\right)$ can be written in the form $\theta'\mathbf{t}\left(g, X\right)$. Consider the first part of the potential

$$\begin{aligned} \sum_i \sum_j g_{ij}u_{ij} &= \sum_i \sum_j g_{ij} \sum_{p=1}^{P} \theta_{up} H_{up}\left(X_i, X_j\right) \\ &= \sum_{p=1}^{P} \theta_{up} \sum_i \sum_j g_{ij} H_{up}\left(X_i, X_j\right) \\ &\equiv \sum_{p=1}^{P} \theta_{up} t_{up}\left(g, X\right) \\ &= \theta'_u \mathbf{t}_u\left(g, X\right) \end{aligned}$$

where $t_{up}\left(g, X\right) \equiv \sum_i \sum_j g_{ij} H_{up}\left(X_i, X_j\right)$, $\theta_u = \left(\theta_{u1}, ..., \theta_{uP}\right)'$ and $\mathbf{t}_u\left(g, X\right) = \left(t_{u1}\left(g, X\right), ..., t_{uP}\left(g, X\right)\right)'$. Analogously define $\theta_m = \left(\theta_{m1}, \theta_{m2}, ..., \theta_{mL}\right)'$ and $\mathbf{t}_m\left(g, X\right) = \left(t_{m1}\left(g, X\right), t_{m2}\left(g, X\right), ..., t_{mL}\left(g, X\right)\right)'$ and $\theta_v = \left(\theta_{v1}, \theta_{v2}, ..., \theta_{vS}\right)'$ and $\mathbf{t}_v\left(g, X\right) = \left(t_{v1}\left(g, X\right), t_{v2}\left(g, X\right), ..., t_{vS}\left(g, X\right)\right)'$. It follows that

$$\begin{aligned} \sum_i \sum_{j>i} g_{ij}g_{ji}m_{ij} &= \sum_i \sum_{j>i} g_{ij}g_{ji} \sum_{l=1}^{L} \theta_{ml} H_{ml}\left(X_i, X_j\right) \\ &= \sum_{l=1}^{L} \theta_{ml} \sum_i \sum_{j>i} g_{ij}g_{ji} H_{ml}\left(X_i, X_j\right) \\ &= \sum_{l=1}^{L} \theta_{ml} t_{ml}\left(g, X\right) \\ &= \theta'_m \mathbf{t}_m\left(g, X\right) \end{aligned}$$

[45] For more details and definitions see Monderer and Shapley (1996).

and

$$\begin{aligned}
\sum_i \sum_j g_{ij} \sum_{k \neq i,j} g_{jk} v_{ij} &= \sum_i \sum_j g_{ij} \sum_{k \neq i,j} g_{jk} \sum_{s=1}^{S} \theta_{vs} H_{vs}(X_i, X_k) \\
&= \sum_{s=1}^{S} \theta_{vs} \sum_i \sum_j g_{ij} \sum_{k \neq i,j} g_{jk} H_{vs}(X_i, X_k) \\
&= \sum_{s=1}^{S} \theta_{vs} t_{vs}(g, X) \\
&= \theta_v' \mathbf{t}_v(g, X)
\end{aligned}$$

Therefore $Q(g, X)$ can be written in the form $\theta' \mathbf{t}(g, X)$, where $\theta = (\theta_u, \theta_m, \theta_v)'$ and $\mathbf{t}(g, X) = [\mathbf{t}_u(g, X), \mathbf{t}_m(g, X), \mathbf{t}_v(g, X)]'$

$$\begin{aligned}
Q(g, X) &= \theta_u' \mathbf{t}_u(g, X) + \theta_m' \mathbf{t}_m(g, X) + \theta_v' \mathbf{t}_v(g, X) \\
&= \theta' \mathbf{t}(g, X)
\end{aligned}$$

and the stationary distribution is

$$\pi(g, X) = \frac{\exp[\theta' \mathbf{t}(g, X)]}{\sum_{\omega \in \mathcal{G}} \exp[\theta' \mathbf{t}(\omega, X)]}.$$

Model without preference shocks: characterization of Nash networks

It is helpful to consider a *special case* of the model, in which there are no preference shocks: the characterization of equilibria and long run behavior for such model provides intuition about the dynamic properties of the full structural model.

Let $\mathcal{N}(g)$ be the set of networks that differ from g by only one element of the matrix, i.e.

$$\mathcal{N}(g) \equiv \{g' : g' = (g_{ij}', g_{-ij}), \text{ for all } g_{ij}' \neq g_{ij}, \text{ for all } i, j \in \mathcal{I}\}. \tag{19}$$

A Nash network is defined as a network in which any player has no profitable deviations from his current linking strategy, when randomly selected from the population. The following results characterize the set of the pure-strategy Nash equilibria and the long run behavior of the model with no shocks.

PROPOSITION 2 ***(Model without Shocks: Equilibria and Long Run)***
Consider the model without idiosyncratic preference shocks. Under Assumptions 1 and 2:

1. *There exists at least one pure-strategy Nash equilibrium network*

2. *The set* $\mathcal{NE}(\mathcal{G}, X, U)$ *of all pure-strategy Nash equilibria of the network formation game is completely characterized by the local maxima of the potential function.*

$$\mathcal{NE}(\mathcal{G}, X, U) = \left\{ g^* : g^* = \arg \max_{g \in \mathcal{N}(g^*)} Q(g, X) \right\} \tag{20}$$

3. *Any pure-strategy Nash equilibrium is an absorbing state.*

4. *As* $t \to \infty$*, the network converges to one of the Nash networks with probability 1.*

Proof. 1) The existence of Nash equilibria follows directly from the fact that the network formation game is a potential game with finite strategy space. (see Monderer and Shapley (1996) for details)
2) The set of Nash equilibria is defined as the set of g^* such that, for every i and for every $g_{ij} \neq g^*_{ij}$

$$U_i\left(g^*_{ij}, g^*_{-ij}, X\right) \geq U_i\left(g_{ij}, g^*_{-ij}, X\right)$$

Therefore, since Q is a potential function, for every $g_{ij} \neq g^*_{ij}$

$$Q\left(g^*_{ij}, g^*_{-ij}, X\right) \geq Q\left(g_{ij}, g^*_{-ij}, X\right)$$

Therefore g^* is a maximizer of Q. The converse is easily checked by the same reasoning.
3) Suppose $g^t = g^*$. Since this is a Nash equilibrium, no player will be willing to change her linking decision when her turn to play comes. Therefore, once the chain reaches a Nash equilibrium, it cannot escape from that state.
4) The probability that the potential will increase from t to $t+1$ is

$$Pr\left[Q\left(g^{t+1}, X\right) \geq Q\left(g^t, X\right)\right] =$$

$$\begin{aligned} &= \sum_i \sum_j Pr\left(m^{t+1} = ij\right) \underbrace{Pr\left[U_i\left(g^{t+1}_{ij}, g^t_{-ij}, X\right) \geq U_i\left(g^t_{ij}, g^t_{-ij}, X\right)\middle| m^{t+1} = ij\right]}_{=1 \text{ because agents play Best Response, conditioning on } m^{t+1}} \\ &= \sum_i \sum_j \rho_{ij} = 1. \end{aligned}$$

By part 3) of the proposition, a Nash network is an absorbing state of the chain. Therefore any probability distribution that puts probability 1 on a Nash network is a stationary distribution. For any initial network, the chain will converge to one of the stationary distributions. It follows that in the long run the model will be in a Nash network, i.e. for any $g^0 \in \mathcal{G}$

$$\lim_{t \to \infty} Pr\left[g^t \in NE \middle| g^0\right] = 1.$$

■

Proof of Theorem 1

1. The sequence of networks $[g^0, g^1, ...]$ generated by the network formation game is a markov chain. Inspection of the transition probability proves that the chain is irreducible and aperiodic, therefore it is ergodic. The existence of a unique stationary distribution then follows from the ergodic theorem (see Gelman et al. (1996) for details).

2. A sufficient condition for stationarity is the *detailed balance* condition. In our case this requires

$$P_{gg'}\pi_g = P_{g'g}\pi_{g'} \tag{21}$$

where

$$\begin{aligned} P_{gg'} &= \Pr\left(g^{t+1} = g' \middle| g^t = g\right) \\ \pi_g &= \pi\left(g^t = g\right) \end{aligned}$$

Notice that the transition from g to g' is possible if these networks differ by only one element g_{ij}. Otherwise the transition probability is zero and the detailed balance condition is satisfied. Let's consider the nonzero probability transitions, with $g = (1, g_{-ij})$ and $g' = (0, g_{-ij})$. Define $\Delta Q \equiv Q(1, g_{-ij}, X) - Q(0, g_{-ij}, X)$.

$$\begin{aligned} P_{gg'}\pi_g &= \Pr\left(m^t = ij\right) Pr\left(g_{ij} = 0 \middle| g_{-ij}\right) \frac{\exp\left[Q(1, g_{-ij}, X)\right]}{\sum\limits_{\omega\in\mathcal{G}} \exp\left[Q(\omega, X)\right]} \\ &= \rho\left(g_{-ij}, X_i, X_j\right) \times \frac{1}{1+\exp\left[\Delta Q\right]} \times \frac{\exp\left[Q(1, g_{-ij}, X) + Q(0, g_{-ij}, X) - Q(0, g_{-ij}, X)\right]}{\sum\limits_{\omega\in\mathcal{G}} \exp\left[Q(\omega, X)\right]} \\ &= \rho\left(g_{-ij}, X_i, X_j\right) \times \frac{1}{1+\exp\left[\Delta Q\right]} \times \frac{\exp\left[Q(1, g_{-ij}, X) - Q(0, g_{-ij}, X)\right]\exp\left[Q(0, g_{-ij}, X)\right]}{\sum\limits_{\omega\in\mathcal{G}} \exp\left[Q(\omega, X)\right]} \\ &= \rho\left(g_{-ij}, X_i, X_j\right) \frac{\exp\left[\Delta Q\right]}{1+\exp\left[\Delta Q\right]} \frac{\exp\left[Q(0, g_{-ij}, X)\right]}{\sum\limits_{\omega\in\mathcal{G}} \exp\left[Q(\omega, X)\right]} \\ &= \Pr\left(m^t = ij\right) \Pr\left(g_{ij} = 1 \middle| g_{-ij}\right) \frac{\exp\left[Q(0, g_{-ij}, X)\right]}{\sum\limits_{\omega\in\mathcal{G}} \exp\left[Q(\omega, X)\right]} \\ &= P_{g'g}\pi_{g'} \end{aligned}$$

So the distribution (5) satisfies the detailed balance condition. Therefore it is a stationary distribution for the network formation model. From part 1) of the proposition, we know that the process is ergodic and it has a unique stationary distribution. Therefore $\pi(g, X)$ is also the unique stationary distribution.